Black Practice Test Solutions

1. Point C must lie somewhere on the horizontal line $y = 10$ since its coordinates are $(k,10)$. Point B is already known to be on that line. Points A and B are both on the vertical line $x = 9$. This means that angle CBA is a right angle and triangle ABC is a right triangle. We know leg AB measures $10 - 3 = 7$ units and we are told that AC measures 25 units. We can use the Pythagorean Theorem to calculate the length of the unknown leg BC. Let m be the measure of BC. We have $7^2 + m^2 = 25^2$ or $49 + m^2 = 625$. Subtracting 49 from both sides, we get $m^2 = 625 - 49$ or $m^2 = 576$. Taking the square root of both sides, we get $m = \pm 24$. (Notice this is the 7-24-25 Pythagorean triple.) This means that point $C(k,10)$ could be 24 units to the left or to the right of point B. Since we are looking for the smallest value for k, we'll go 24 units to the left, which gives us $k = 9 - 24 = $ **–15**.

2. If a perpendicular line is drawn vertically down the center of the rectangle, a small triangle is separated from the shaded region. This triangle can then be rotated 180° around the center of the rectangle, and it will then occupy a congruent unshaded region. When this rotation occurs, the new shaded area is a rectangle which is one-sixth the area of the entire rectangle.

If the side length of the original rectangle is s, then the area of the shaded rectangle is $\frac{s}{2} \times \frac{s}{3} = \frac{s^2}{6} = 24\,\text{in}^2$. Consequently, $s^2 = 144$, and $s = 12\,\text{in}$. The perimeter of the original square, then, is $4 \times 12 = 48\,\text{in}$.

3. The area of square ABCD is $36\,\text{cm}^2$, so $\text{AB} = 6\,\text{cm}$. Likewise, $\text{DC} = 6\,\text{cm}$, and because $\text{ND} = \frac{1}{2}\text{CN}$, $\text{ND} = 2\,\text{cm}$ and $\text{CN} = 4\,\text{cm}$. Further, the area of NFMD is $18\,\text{cm}^2$, so $(\text{ND})(\text{DM}) = 2(\text{DM}) = 18$ and $\text{DM} = 9\,\text{cm}$.

The perimeter of ABEM, then, is $\text{AB} + \text{BC} + \text{CE} + \text{EF} + \text{FM} + \text{MD} + \text{DA} = 6 + 6 + 9 + 4 + 2 + 9 + 6 = 42\,\text{cm}$.

4. What are we asked to find? The area of $\triangle$ ABC.

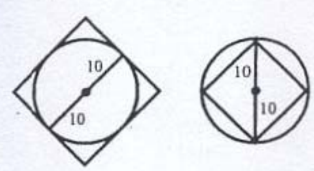
To determine the area of a triangle, the base and height of the triangle must be found. The information provided about the areas of the circles can be used to determine the radius of each circle, and adding the radii will determine the lengths of the sides of the triangle. The Pythagorean theorem can then be used to calculate the height of the triangle, and from that the area can be calculated.

Circle A has area 16π, and the formula for the area of a circle is πr^2. Hence, the radius of circle A is 4 cm. Likewise, the radius of circle C is also 4 cm, and the radius of circle B is 1 cm. $\triangle$ ABC is isosceles with congruent sides of length 5 cm and base of length 8 cm. The height from vertex B forms two right triangles with hypotenuse 5 cm and leg 4 cm. The height, then, is $\sqrt{5^2 - 4^2} = 3\,\text{cm}$.

The area of $\triangle$ ABC, then, is $A = \frac{1}{2}bh = \frac{1}{2}(8)(3) = 12\,\text{cm}^2$.

Does the answer make sense? Yes. The area of circle A is $16\pi\,\text{cm}^2$, or approximately $50\,\text{cm}^2$. By visual comparison, it seems reasonable that the area of $\triangle$ ABC is roughly one-fourth the area of circle A.

5. The area of the region between the two squares is the positive difference between the areas of the squares. If the radius of the circle is 10 centimeters, the diameter is 20 centimeters. The diameter of the circle is equal to the side length of the larger square and the diagonal of the smaller square. The larger square has an area of $20 \times 20 = 400$ square centimeters. Since the smaller square has a diagonal of 20 cm, its area is $1/2 \times 20 \times 20 = 200$. (A square is a rhombus with equal diagonals, so we could use the formula for the area of a rhombus: $(1/2)\, d_1 d_2$.) The number of square centimeters in the area between the two squares is then $400 - 200 = $ **200** square centimeters.



6. You've known for a long time that a circle with diameter 16 inches has a radius of 8 inches and an area of $\pi \cdot 8^2 = 64\pi$. And a circle with diameter 8 inches has area $\pi \cdot 4^2 = 16\pi$. Finding the areas of the circles in this problem isn't the hard part—the hard part is adding them all together!

The sum of all the areas can be written as an infinite series, that is, as the sum of an infinity of terms. That sum is

$$64\pi + 16\pi + 4\pi + \pi + \tfrac{1}{4}\pi + \tfrac{1}{16}\pi + \tfrac{1}{64}\pi + \cdots$$

Adding the first four terms is easy: $64\pi + 16\pi + 4\pi + \pi = 85\pi$. Adding the fractions isn't quite as easy, especially since there are infinitely many of them, but there's a cool trick for finding the sum of an infinite series.

Call the sum S such that $S = \frac{1}{4}\pi + \frac{1}{16}\pi + \frac{1}{64}\pi + \frac{1}{256}\pi + \cdots$. Then, by multiplying both sides by 1/4 and subtracting from the original equation, we get the following result:

$$\begin{aligned} S &= \tfrac{1}{4}\pi + \tfrac{1}{16}\pi + \tfrac{1}{64}\pi + \tfrac{1}{256}\pi + \cdots \\ -\tfrac{1}{4}S &= \qquad\quad \tfrac{1}{16}\pi + \tfrac{1}{64}\pi + \tfrac{1}{256}\pi + \cdots \\ \hline \tfrac{3}{4}S &= \tfrac{1}{4}\pi \end{aligned}$$

When subtracting, all of the terms on the right side of the equation, except $\frac{1}{4}\pi$, cancel out. The algebra is now easy. If $\frac{3}{4}S = \frac{1}{4}\pi$, then $S = \frac{4}{3} \times \frac{1}{4}\pi = \frac{1}{3}\pi$.

Therefore, the total area of all circles is $85\pi + \frac{1}{3}\pi$, which is approximately **268 square inches.**

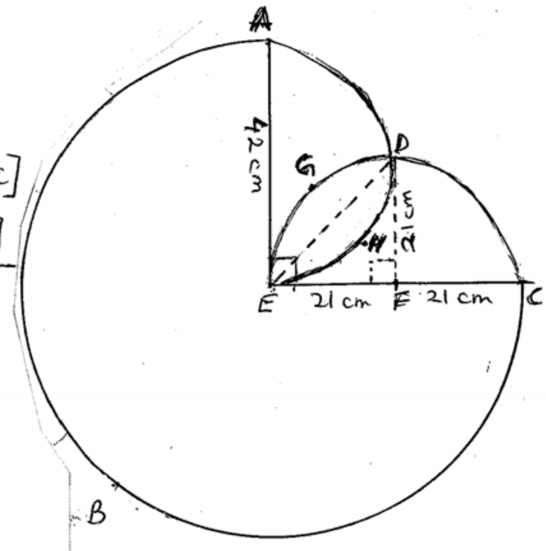
7. Answer: 5292 cm^2

Area of the figure

$= A[\frac{3}{4} \odot ABC]$

$+ A[\text{semi} \odot ADE] + A[\text{semi} \odot EDC]$

$- A[\text{sector } EGD] - A[\text{sector } EHD]$



$A[\frac{3}{4} \odot ABC]$

$= \frac{3}{4} \cdot \pi \cdot r^2$

$= \frac{3}{4} \cdot \frac{22}{7} \cdot 42 \cdot 42$

$= \boxed{4158\ cm^2}$

$A[\text{semi} \odot ADE] = A[\text{semi} \odot EDC] = \frac{1}{2}\pi r^2 = \frac{1}{2} \cdot \frac{22}{7} \cdot 21 \cdot 21$

$= \boxed{693\ cm^2}$

$A[\text{sector } EGD] = A[\text{sector } EHD]$

$= A[\frac{1}{4} \odot EGD] - A[\Delta EFD]$

$= \frac{1}{4}\pi r^2 - \frac{1}{2} \cdot b \cdot h$

$= \frac{1}{4} \cdot \frac{22}{7} \cdot 21 \cdot 21 - \frac{1}{2} \cdot 21 \cdot 21$

$= \frac{693}{2} - \frac{441}{2} = \frac{252}{2} = \boxed{126\ cm^2}$

Area of the figure

$= 4158\ cm^2 + 693\ cm^2 + 693\ cm^2 - 126\ cm^2 - 126\ cm^2$

$= \boxed{5292\ cm^2}$

8. *Leading by a Head.* Assume that the earth is spherical and has radius R feet. Then the feet will travel $2\pi R$ feet. The head will travel $2\pi(R + 6)$ feet. The difference is $2\pi(6) = 12\pi$ feet.

9. *Solution:* Let r be the radius of the circle. Notice ΔOAB is a 45°, 45° right triangle. Hence, one leg is $r\frac{\sqrt{2}}{2}$, which means the length of the side of the square is $r\sqrt{2}$. The area of the square is $2r^2$. The area of the circle is πr^2. The ratio of the areas is $2 : \pi$.

10. **Possible Solutions**

Part 1:
In one year, a person produces 365 pints of gas. This is equivalent to $\frac{365}{8} \bullet 231 \text{ in}^3$

If time t is measured in days then in general, we can solve this problem by solving for the radius in the following equation:

$$(1) \quad \frac{t}{8} \bullet 231 = \frac{4}{3} \bullet (3.14) \bullet r^3(1)$$

Solving for r yields

$$(2) \quad r = \sqrt[3]{\frac{693t}{100.48}} = \sqrt[3]{6.9t}$$

If t =365, then the radius will be about 13.6 inches and so the diameter is 27.2 inches.

Part 2:
Here we need to calculate a time and so we must solve equation (1) for t:

$$(3) \quad t = 0.45r^3$$

Since half the diameter implies half the radius it follows that r = 6.8. Using equation (3) it follows that t = 45.6 days. This may be very surprising to many students if they apply proportional thinking to diameter and time directly. Halving the diameter will decrease the volume to one eighth $(\frac{1}{2})^3$ that after one year. Then 365/8 = 45.6 as well. This means that the diameter would be half sometime during February 15.

11. *General Painted Dice.* In general, the probability is $1/N$. The total number of faces painted is $6N^2$, and the total number of faces on the small cubes is $6N^3$, so the probability of getting a painted face is $6N^2/6N^3 = 1/N$.

12. *Prisms in a Box.* Six prisms. A 3″ × 3″ × 3″ prism has volume of 9 cubic inches. A 4″ × 4″ × 4″ prism has volume of 64 cubic inches. So no more than seven of the smaller prisms can fit inside the larger prism. If you stack four of the smaller prisms, you can fill a 3″ × 3″ × 4″ space. There remains room for only two more pieces to be stacked in the larger prism.

13. If the edge of a cube is x inches, then the diagonal of each face is $x\sqrt{2}$. The space diagonal is the hypotenuse of a right triangle with legs of length x and $x\sqrt{2}$, so the space diagonal is $x\sqrt{3}$. The space diagonal is $3\sqrt{3}$, so $x = 3$, and the area of each face is $3 \times 3 = 9\,\text{in}^2$. The surface area of the entire cube is $6 \times 9 = 54\,\text{in}^2$.